Suggested solutions to the Contract Theory exam on Jan. 8, 2014
VERSION: 30 January 2014

Question 1 (adverse selection)

Part a)

- First note that the question states that $\varphi' > 0$, which should be understood as φ being strictly increasing for all values of the argument. However, in applications it might be useful to assume, for example, a quadratic function, $\varphi(x) = \frac{1}{2}x^2$, in which case we would have $\varphi'(x) > 0$ for all $x > 0$ but $\varphi'(0) = 0$. The solution presented below will deal also with the more general case that allows for $\varphi'(0) = 0$ (although this amounts to answering more than is actually asked about in the question).

- Given that the two types are offered different contracts, the government's problem is: Choose $\underline{t}, \underline{q}, \bar{t}, \bar{q}$ so as to maximize:[1]

$$\boxed{V = \nu\left(\underline{t} - \underline{\theta}\underline{q}\right) + (1-\nu)\left(\bar{t} - \bar{\theta}\bar{q}\right) - \varphi\left[\left|\left(\underline{t} - \underline{\theta}\underline{q}\right) - \left(\bar{t} - \bar{\theta}\bar{q}\right)\right|\right],}$$

subject to the two IC-constraints and the budget constraint.

$$\boxed{\underline{t} - \underline{\theta}\underline{q} \geq \bar{t} - \underline{\theta}\bar{q},} \qquad \text{(IC-good)}$$

$$\boxed{\bar{t} - \bar{\theta}\bar{q} \geq \underline{t} - \bar{\theta}\underline{q},} \qquad \text{(IC-bad)}$$

$$\boxed{\nu S\left(\underline{q}\right) + (1-\nu) S\left(\bar{q}\right) \geq \nu\underline{t} + (1-\nu)\bar{t}.} \qquad \text{(budget)}$$

- The function φ depends on the absolute value of the utility difference between the high ability type and the low-ability type. It is useful to note that this difference must be strictly positive whenever IC-good is satisfied. That is, IC-good implies $\left(\underline{t} - \underline{\theta}\underline{q}\right) > \left(\bar{t} - \bar{\theta}\bar{q}\right)$. This is useful since it means that we can ignore the absolute value signs when solving the problem. Later on we will also use the result that the inequality is strict.

  - Proof of claim: Suppose not, so that we have $\left(\underline{t} - \underline{\theta}\underline{q}\right) \leq \left(\bar{t} - \bar{\theta}\bar{q}\right)$. This inequality together with IC-good implies that

$$\bar{t} - \underline{\theta}\bar{q} \overbrace{\leq}^{\text{By IC-good}} \underline{t} - \underline{\theta}\underline{q} \leq \bar{t} - \bar{\theta}\bar{q}$$

[1] The method for solving the problem that is described below uses a Lagrangian and does not make use of the suggestion in the question that we may assume that the budget constraint is binding at the optimum. An alternative method, which also is fine, is to indeed make use of that suggestion, using the binding budget constraint to make a substitution in the objective function. One would also use the other suggestion in the question and assume that IC-bad does not bind at the optimum. Thereafter one can argue that the remaining constraint, IC-good, must bind at the optimum and substitute in that one too in the objective function. One can also argue that the absolute value signs do not matter (compare the arguments below). Thus the resulting reduced-form objective function can easily be differentiated to obtain the first-order conditions that define the optimal second-best quantities.

or

$$\overline{t} - \underline{\theta}\overline{q} \leq \overline{t} - \overline{\theta}\overline{q},$$

which can be rewritten as $\left(\overline{\theta} - \underline{\theta}\right)\overline{q} \leq 0$, which is impossible given the assumptions that $\overline{\theta} > \underline{\theta}$ and $\overline{q} > 0$. This proves the claim.

- We are also, according to the question, allowed to assume that IC-bad does not bind at the optimum. We can now rewrite the problem as: Choose $\underline{t}, \underline{q}, \overline{t}, \overline{q}$ so as to maximize:

$$\boxed{V = \nu\left(\underline{t} - \underline{\theta}\underline{q}\right) + (1-\nu)\left(\overline{t} - \overline{\theta}\overline{q}\right) - \varphi\left[\left(\underline{t} - \underline{\theta}\underline{q}\right) - \left(\overline{t} - \overline{\theta}\overline{q}\right)\right],}$$

subject to IC-good and the budget constraint.

$$\boxed{\underline{t} - \underline{\theta}\underline{q} \geq \overline{t} - \underline{\theta}\overline{q},} \qquad \text{(IC-good)}$$

$$\boxed{\nu S\left(\underline{q}\right) + (1-\nu) S\left(\overline{q}\right) \geq \nu\underline{t} + (1-\nu)\overline{t}.} \qquad \text{(budget)}$$

- The Lagrangian is:

$$\begin{aligned}\mathcal{L} \;=\; & \nu\left(\underline{t} - \underline{\theta}\underline{q}\right) + (1-\nu)\left(\overline{t} - \overline{\theta}\overline{q}\right) - \varphi\left[\left(\underline{t} - \underline{\theta}\underline{q}\right) - \left(\overline{t} - \overline{\theta}\overline{q}\right)\right] \\ & +\mu\left[\nu S\left(\underline{q}\right) + (1-\nu) S\left(\overline{q}\right) - \nu\underline{t} - (1-\nu)\overline{t}\right] \\ & +\lambda\left[\underline{t} - \underline{\theta}\underline{q} - \overline{t} + \underline{\theta}\overline{q}\right].\end{aligned}$$

- **FOC w.r.t. $\underline{t}$:**

$$\frac{\partial\mathcal{L}}{\partial\underline{t}} = 0 \Leftrightarrow \nu - \varphi'\left[\left(\underline{t} - \underline{\theta}\underline{q}\right) - \left(\overline{t} - \overline{\theta}\overline{q}\right)\right] = \nu\mu - \lambda. \qquad (1)$$

- **FOC w.r.t. $\underline{q}$:**

$$\frac{\partial\mathcal{L}}{\partial\underline{q}} = 0 \Leftrightarrow \nu\underline{\theta} - \underline{\theta}\varphi'\left[\left(\underline{t} - \underline{\theta}\underline{q}\right) - \left(\overline{t} - \overline{\theta}\overline{q}\right)\right] = \nu\mu S'\left(\underline{q}\right) - \lambda\underline{\theta}. \qquad (2)$$

- **FOC w.r.t. $\overline{t}$:**

$$\frac{\partial\mathcal{L}}{\partial\overline{t}} = 0 \Leftrightarrow (1-\nu) + \varphi'\left[\left(\underline{t} - \underline{\theta}\underline{q}\right) - \left(\overline{t} - \overline{\theta}\overline{q}\right)\right] = (1-\nu)\mu + \lambda. \qquad (3)$$

- **FOC w.r.t. $\overline{q}$:**

$$\frac{\partial\mathcal{L}}{\partial\overline{q}} = 0 \Leftrightarrow (1-\nu)\overline{\theta} + \overline{\theta}\varphi'\left[\left(\underline{t} - \underline{\theta}\underline{q}\right) - \left(\overline{t} - \overline{\theta}\overline{q}\right)\right] = (1-\nu)\mu S'\left(\overline{q}\right) + \lambda\underline{\theta}. \quad (4)$$

- By adding (1) and (3) we have[2]

$$\mu = 1.$$

This implies, in particular, that the budget constraint binds (we know that since $\mu > 0$). We were actually allowed to assume that, but this help is not of much use here as I will make use of the more specific result $\mu = 1$ in the calculations below.

[2] Here and in a few other places below I'm fairly brief — it's good if the student shows all the steps in his/her calculations. One reason for that is that then it is possible to give them some credit for their answers even if there is some sloppy error in the calculations.

- Also, (1) and (2), together with $\mu = 1$, imply

$$S'\left(\underline{q}^{SB}\right) = \underline{\theta}. \tag{5}$$

This equality is the condition that defines the able type's first-best quantity, so we have the result that the able type's quantity is not distorted under second best: there is "efficiency at the top".

- Further, (1) and $\mu = 1$ give us

$$\lambda = \varphi' \left[\left(\underline{t} - \underline{\theta}\underline{q}^{SB} \right) - \left(\bar{t} - \overline{\theta}\overline{q}^{SB} \right) \right]. \tag{6}$$

- Finally, (4) together with $\mu = 1$ and (6) lead to

$$\begin{aligned} & (1-\nu)\,\overline{\theta} + \overline{\theta}\varphi' \left[\left(\underline{t} - \underline{\theta}\underline{q}^{SB} \right) - \left(\bar{t} - \overline{\theta}\overline{q}^{SB} \right) \right] \\ = \quad & (1-\nu)\, S'\left(\overline{q}^{SB}\right) + \underline{\theta}\varphi' \left[\left(\underline{t} - \underline{\theta}\underline{q}^{SB} \right) - \left(\bar{t} - \overline{\theta}\overline{q}^{SB} \right) \right] \end{aligned}$$

or

$$S'\left(\overline{q}^{SB}\right) = \overline{\theta} + \frac{\left(\overline{\theta} - \underline{\theta}\right) \varphi' \left[\left(\underline{t} - \underline{\theta}\underline{q}^{SB} \right) - \left(\bar{t} - \overline{\theta}\overline{q}^{SB} \right) \right]}{1-\nu}.$$

- If the last term on the right-hand side is zero, then also the "not able" type's quantity is at its first best level. However:
  - we have assumed that $\overline{\theta} - \underline{\theta} > 0$, $1 - \nu > 0$, and $\varphi' > 0$,
  - and we have previously shown that $\left(\underline{t} - \underline{\theta}\underline{q}^{SB} \right) > \left(\bar{t} - \overline{\theta}\overline{q}^{SB} \right)$ [this second argument is not needed if indeed $\varphi'(x) > 0$ for all x, but it would be needed if allowing for a function for which $\varphi'(0) = 0$].

- Therefore, the right-hand side is strictly larger than $\overline{\theta}$, which means that the "not able" type's quantity is distorted downwards ($\overline{q}^{SB} < \overline{q}^{FB}$) relative to its first best level.

Part b)

- The government trades off efficiency and equality — given asymmetric information, it cannot get both. The reason for this is that if the government chose contracts implying full efficiency and full equality (given that the types choose the contracts directed at them), then the able type would have an incentive to pick the "not able" type's contract — the able type's incentive compatibility constraint would be violated. In order to satisfy incentive compatibility, the government can in principle adjust either the required work loads (i.e. deviating from efficiency) or allow more inequality. In the choice between adjusting the able type's quantity and the "not able" type's quantity, the government always prefers to adjust only the latter. The intuitive reason is that the able type is (since he is more productive) more useful in producing resources (which later can be redistributed to the other agent, if the government so wishes); it therefore would be a waste not to let the able agent produce the efficient quantity. By letting the "not able" type produce less than his first-best quantity, the government needs to transfer less money to him in order to achieve a given utility level of that agent type. Therefore the "not able" type's contract becomes less attractive for the able type, and so IC-good becomes easier to satisfy. This reasoning suggests that if the governments wants to deviate from full efficiency, then it will do that by adjusting only the "not able" type's quantity (and to do that downwards), leading to $\underline{q}^{SB} = \underline{q}^{FB}$ and $\overline{q}^{SB} < \overline{q}^{FB}$. But couldn't it be that the government prefers to compromise only with equality, keeping full efficiency? One can understand intuitively why that will not be the case. The reason is that the cost in terms of lost production of letting the "not able" type's quantity be slightly below its first best level is negligible: Since the quantity $\overline{q}^{FB}$ is optimally chosen (meaning that the gradient of the surplus function is zero at $\overline{q}^{FB}$), reducing it slightly leads to a loss that is of "second order" magnitude only.

Part c), (i)

- We used this assumption when checking that the second-order condition of the problem was satisfied.

Part c), (ii)

- The implication of the assumption is that the "informational cost function" is convex. If that cost function were concave, it would mean that there were economies of scale in terms of the informational costs of letting the bad type agent choose a relatively high quantity.

- The "informational cost function" tells us how much extra money the principal, if asking the bad type to be active and produce a particular quantity, must pay the good type in order to make sure that IC-good is satisfied. This amount of money is "extra" in the sense that it comes on top of the money that the good type gets as a compensation for her production costs.

Question 2 (moral hazard)

Part a)

- We make use of the hint that, since $\beta < 1$, the king's top priority is rent extraction. Since the king can dictate to the farmers exactly what effort level they must choose, the king does not need to use the tax τ and the income support b as instruments to appropriately incentivize the farmers. Instead the king should set τ and b such that the king's rents are maximized. First, b is set as low as possible, namely so that LL-low binds:

$$\boxed{b^{FB} = 0.}$$

  Second, the tax τ is set as large as possible, namely so that LL-high binds:

$$\boxed{\tau^{FB} = w.}$$

- What about the effort level?

  a) The fact that the king's top priority is rent extraction means that *one possibility* is that the effort is chosen to be as large as possible, so $\boxed{e = 1.}$ The reason is that this effort level maximizes the amount of resources that the king can collect through the tax τ.

  b) *Another possibility* is that the king prefers a somewhat lower effort level than $e = 1$. The reason for this would be that the king also, even if rent extraction is important, cares about the farmers' expected utility (see the last term in the objective), which includes the effort cost $\varphi(e)$. In particular, we should expect that setting $\boxed{e < 1}$ is optimal for the king if the weight β is sufficiently large (but still satisfying $\beta < 1$).

- The students are getting credit for answering something along the lines of either a) or b).[3]

Part b)

- This is a moral hazard problem with an agent who is risk neutral and protected by limited liability. The agent chooses an effort level from a continuum and there are only two possible outcomes (high and low income).

- We can write the principal's problem as follows: Choose τ, b and e so as to maximize

$$\begin{aligned}
V &= e\tau - (1-e)\,b - \alpha\,|\overline{u} - \underline{u}| + \beta E[u] \\
&= e\tau - (1-e)\,b - \alpha\left|\left(w - \tau - \frac{1}{2}e^2\right) - \left(b - \frac{1}{2}e^2\right)\right| \\
&\quad + \beta\left[e(w-\tau) + (1-e)\,b - \frac{1}{2}e^2\right],
\end{aligned}$$

[3] By solving the problem formally, which is not required and does not give any credit, one can show that the first best effort level is $e^{FB} = \min\left\{\frac{w}{\beta}, 1\right\}$.

subject to

$$\begin{aligned} & e\,(w-\tau)+(1-e)\,b-\frac{1}{2}e^2 \\ \geq \quad & e'\,(w-\tau)+(1-e')\,b-\frac{1}{2}\,(e')^2 \qquad \text{for all } e' \geq 0, \qquad \text{(IC)} \end{aligned}$$

$$e\tau-(1-e)\,b \geq 0, \qquad \text{(budget)}$$

$$\tau \leq w, \qquad \text{(LL-high)}$$

$$b \geq 0. \qquad \text{(LL-low)}$$

- When solving the problem we can make use of the so-called first-order approach. This amounts to replacing the infinitely many IC constraints with one single condition, namely the agent's first-order condition. Thus, consider the agent's effort choice problem, given some τ and b:

$$\max_{e\in[0,1]}\left\{e\,(w-\tau)+(1-e)\,b-\frac{1}{2}e^2\right\}.$$

The first-order condition is:

$$w-\tau-b=e \Leftrightarrow \tau=w-e-b. \qquad \text{(IC-new)}$$

- Let us guess that, at the optimum, $\overline{u} \geq \underline{u}$ (we must verify this later, after having found a candidate solution). We can then rewrite the principal's objective as follows:

$$V=e\tau-(1-e)\,b-\alpha\,(w-\tau-b)+\beta\left[e\,(w-\tau-b)+b-\frac{1}{2}e^2\right].$$

Plugging in $\tau=w-e-b$ from IC-new in the objective function, we obtain

$$\begin{aligned} V &= e\,(w-e-b)-(1-e)\,b-\alpha e+\beta\left[e^2+b-\frac{1}{2}e^2\right] \\ &= e\,(w-\alpha)-\frac{(2-\beta)}{2}e^2-(1-\beta)\,b. \qquad (7) \end{aligned}$$

Moreover, given (IC-new), the (LL-high) constraint now becomes

$$w-e-b \leq w \Leftrightarrow e+b \geq 0. \qquad \text{(LL-high-new)}$$

But (LL-high-new) is implied by (LL-low), which means that we can ignore (LL-high-new). Finally, the budget constraint can, given (IC-new), be written as

$$e\,(w-e-b)-(1-e)\,b \geq 0 \Leftrightarrow e\,(w-e)-b \geq 0. \qquad \text{(budget-new)}$$

- The new problem amounts to maximizing the objective in (7) with respect to e and b, subject to the constraints (budget-new) and (LL-low). It is quite clear that (LL-low) must bind, meaning that $b=0$. Why is that? The objective in (7) is decreasing in b (recall that $\beta<1$). Moreover, the constraint (budget-new) is relaxed if lowering b. The only constraint that becomes more stringent as we lower b is (LL-low), which therefore must be binding.

- Given that $b = 0$, the principal's problem now simplifies further to: Choose e so as to maximize

$$V = e(w - \alpha) - \frac{(2-\beta)}{2} e^2, \tag{8}$$

subject to

$$e(w - e) \geq 0.$$

- It is easy to verify that the objective in (8), which is strictly concave, equals zero for $e = 0$ and $e = \frac{2(w-\alpha)}{2-\beta}$. Moreover, it has a global optimum at $e = \frac{w-\alpha}{2-\beta}$, which satisfies the constraint $e(w-e) \geq 0$ due to the assumptions that $\alpha < w$ and $\beta < 1$.

- We thus have that the (candidate) second-best optimal effort level is

$$\boxed{e^{SB} = \tfrac{w-\alpha}{2-\beta}.}$$

The (candidate) second-best optimal level of income support is

$$\boxed{b^{SB} = 0.}$$

And, by using IC-new, we have that the (candidate) second-best optimal tax level is

$$\tau^{SB} = w - e^{SB} - b^{SB} = w - \frac{w-\alpha}{2-\beta}$$

or

$$\boxed{\tau^{SB} = \tfrac{(1-\beta)w+\alpha}{2-\beta}.}$$

- We must finally check that, at the candidate optimum, we have indeed $\overline{u} \geq \underline{u}$, as we guessed at the beginning. We have

$$\overline{u}^{SB} \geq \underline{u}^{SB} \Leftrightarrow w - \tau^{SB} - \frac{1}{2}\left(e^{SB}\right)^2 \geq b^{SB} - \frac{1}{2}\left(e^{SB}\right)^2 \Leftrightarrow w \geq \tau^{SB},$$

which indeed holds. This means that the above solutions are not only the candidate solutions but the actual ones.

Part c)
From the lecture slides:

- A **subjective performance evaluation** effectively means that (in Prendergast's words):
  - "Pay is at the discretion of the impressions of a superior".
- Potential advantage with such a measure:
  - It allows **a more holistic view**: an activity is rewarded only if that activity was warranted, given the situation.
- Potential drawback:

  - A subjective measure **can be manipulated or distorted** from its true value.

- Possible ways in which the distortion can manifest itself (these are the three kinds of drawbacks asked about in the question):
  - Theft.
  - Compression of ratings.
  - Rent-seeking activities.

Distortion due to theft

- If "pay is at the discretion of the impressions of a superior"...
  - then it will be tempting for an employer to underreport on performance in order to save on wages.

- Cheatham, Davis, and Cheatham (1996):
  - Sometimes actors are paid on the "net profits" of a movie.

- This practice creates an incentive for the movie company to do "creative accounting" and try to keep the official net profits low.
  - This has led to many court cases.

Distortion due to compression of ratings

- Many empirical studies show that:
  - There is a tendency for the supervisors who evaluate **not to differentiate fully between different performances**.

- Two ways in which supervisors can compress the evaluations:
  - A **centrality bias** (very good and very bad evaluations avoided).
  - A **leniency bias** (poor performers getting too good evaluations).

- Possible reason for this distortion:
  - It is **unpleasant** for the supervisor to hand out a bad evaluation.

- The studies also suggest that the problem is particularly severe **when evaluations are important for pay setting**.
  - Hence some firms separate pay setting from evaluations.

- The overall implication and conclusion:
  - The value of using subjective assessments as a way of providing incentives is reduced. Maybe used **only for training purposes**.

Distortion due to rent-seeking activities

- If "pay is at the discretion of the impressions of a superior"...
  - It will be tempting for an employee to do rent-seeking.
- **"Rent-seeking activities"** refer to:
  - *Any actions that agents carry out that are designed to increase the likelihood of better ratings from supervisors, but have less value on surplus than some alternative activity they could do.*
- Rent-seeking activities can lead to two possible distortions:
  - Resources are wasted because employees spend time and effort to try to influence rather than do their actual jobs.
  - The principal fails to acquire useful information about which employees to promote or reward.
- A study by Bjerke et al. (1987):
  - Supervisors in the U.S. Navy admitted to distorting performance ratings to increase the prospects of their preferred subordinates.

Summing up

- Many jobs are complex and involve **multiple tasks**.
  - Some tasks can be **hard to contract on**.
  - This could lead to a bad outcome — the employee **performing only on the tasks that are rewarded**.
- One potential way of fixing this problem:
  - **Subjective performance evaluation**.
- However, this may also involve problems:
  - **Theft.**
  - **Compression of ratings.**
  - **Rent-seeking activities.**
- The only solution, in some environments, might be to avoid both piece rates and subjective performance evaluations.
  - Instead using **only a fixed rate**.